# Wave-function method used to study the Bethe coupler

N. Marinescu
*Department of Physics, Bucharest University, P.O. Box MG-6, Magurele-Bucharest, Romania*
(Received 16 November 1995; revised manuscript received 4 March 1996)

The Bethe coupler is reinvestigated by use of the wavefunction method for the propagation of the electromagnetic waves in waveguides. The directivity of the secondary guide and the coupling factor are derived by computing the transmission coefficients through the hole of the coupler. A comparison is made with the standard results. [S1063-651X(96)11808-0]

PACS number(s): 42.65.Wi, 84.40.Az, 73.40.Gk

## I. INTRODUCTION

It is well known that the Bethe coupler [1,2] consists of two superposed identical waveguides of uniform rectangular cross section, which communicate with each other by a small cylindrical hole, bored in their common wall. Usually the main guide is operated in the $TE_{01}$ mode and the problem is to compute the power released at the ends of the secondary guide. It has been shown that the Bethe coupler has directional properties, i.e., the forward and backward powers $P_f$ and $P_b$, respectively, in the secondary guide are distinct from each other, and the directivity

$$D=10\ \log_{10}\frac{P_f}{P_b} \tag{1}$$

reaches a maximum for a certain value of the angle $\theta$ between the axes of the two guides. The reason for this $\theta$ dependence of the directivity resides in that an anisotropic $TE_{11}$ is also excited in the cylindrical hole, besides the isotropic $TM_{10}$ mode. In addition, a related parameter defining the quality of the coupler is the coupling factor $C$, defined as

$$C=10\ \log_{10}\frac{P_i}{P_f}, \tag{2}$$

where $P_i$ is the input power entering the main guide.

Usually, the directivity and the coupling of the Bethe coupler is computed by assuming that the power is injected into the secondary guide by diffraction of the radiation by the small, coupling hole. As is well known, Lord Rayleigh's theory of diffraction for short wavelengths [3] has been developed by Bethe [1,2] into an approximate vectorial theory for the diffraction by the coupling hole [4,5]. Recently, a method has been introduced [6,9] that allows one to study the transmission of electromagnetic radiation through waveguides in much the same manner as the wave tunneling through a potential barrier. This method, is not restricted to radiation wavelengths much shorter than the hole size, and recently it has been applied successfully in describing the propagation of the electromagnetic waves through waveguides by analogy with the quantum tunneling phenomenon [10–12]. Apart from being free of the above-mentioned restriction, this method, which we may call the wave-function method, is also useful by its simplicity and flexibility in dealing with various geometries encountered in the physics of the microwaves guides. The wave-function method is used in the present paper to compute the transmission coefficients through the hole of the Bethe coupler for the two modes $TE_{11}$ and $TM_{10}$. The directivity and the coupling factor are then readily obtained, and the results are compared with those of the usual approach.

In the next section the wave-function method is briefly outlined and the transmission coefficient of a waveguide is derived. The results are applied to the Bethe coupler in Sec. III and the directivity and the coupling are discussed in the last section.

## II. THE WAVE-FUNCTION METHOD

It is well known that the propagation of the electromagnetic waves through a waveguide proceeds by two types of transverse standing modes, TE and TM. In the former case the fields are given by the $H_z$ component of the magntetic field along the guide axis

$$H_z=f(x,y)e^{i(kz-\omega_0 t)}, \tag{3}$$

where

$$(\Delta_{x,y}+\kappa^2)f=0,$$

$$(\partial f/\partial \mathbf{n})_\Gamma=0,$$

$\mathbf{n}$ being the vector normal to the $\Gamma$ contour of the cross section, the wave vectors $k$ and $\kappa$ being related to the frequency $\omega$ by

$$\omega^2=c^2k^2+\omega_0^2, \qquad \omega_0=c\kappa, \tag{4}$$

with $c$ denoting the light velocity. Usually, the function $f(x,y)$ is normalized over the cross section of the guide,

$$\int |f(x,y)|^2 dS=1\ .$$

 

For the TM waves the magnetic field $H_z$ is replaced in (3) by the electric field $E_z$, and the boundary condition is $f|_\Gamma = 0$.

The plane wave along the $z$ direction expressed by (3) may be reflected or absorbed by, or transmitted through, various small objects or media placed inside the waveguide, as well as variations of the cross section, which amount to connecting two or more waveguides, resonant cavities, etc. It may also be diffracted by small holes bored in the guide walls, etc. In all these cases we are interested in the amplitude $a_k$ of the plane wave, such that we are led to introduce the wave function [8,9]

$$\Psi(z,t) = \frac{1}{\sqrt{2\omega}} a_k e^{i(kz-\omega t)} \tag{5}$$

by

$$H_z = f(x,y)\sqrt{8\pi}\omega_0 \Psi(z,t) \tag{6}$$

for the TE modes, and a similar relationship for $E_z$ in the case of the TM modes. It has been shown [8,9] in this case that the density of energy (per unit length) is given by

$$\omega|a_k|^2 = 2\omega^2|\Psi|^2 \tag{7}$$

and the energy flux is

$$S = cn\omega|a_k|^2, \tag{8}$$

where

$$n = (1-\omega_0^2/\omega^2)^{1/2}$$

is the refractive index of the guide; i.e., the energy is transported with the group velocity $cn$. In addition, the wave-function given by (5) satisfies a Klein-Gordon type equation in $1+1$ dimensions according to (4), and a density and a current of ''plane waves'' of positive frequencies

$$\omega = (c^2k^2+\omega_0^2)^{1/2}$$

can be defined, which satisfy the continuity equation [8]. As one can see, the whole picture shares many essential features with quantum physics [8,9].

In order to illustrate how the method works we apply it here to the computing of the transmission coefficient of a waveguide. Suppose that a plane wave of wave vector $k$ is sent along a waveguide of length $l$, ranging from $z=0$ to $z=l$. We assume that the lowest frequency transverse mode allowed by the waveguide has such a high frequency $\omega_0$ that the wave vector along the guide axis is imaginary, and denote it by $i\kappa$. We have therefore

$$k^2 = -\kappa^2 + \omega_0^2/c^2. \tag{9}$$

The wave function is given by

$$\Psi(z) = \begin{cases} e^{ikz} + Re^{-ikz}, & z<0 \\ Ae^{\kappa z} + Be^{-\kappa z}, & 0<z<l \\ Te^{ikz}, & l<z, \end{cases} \tag{10}$$

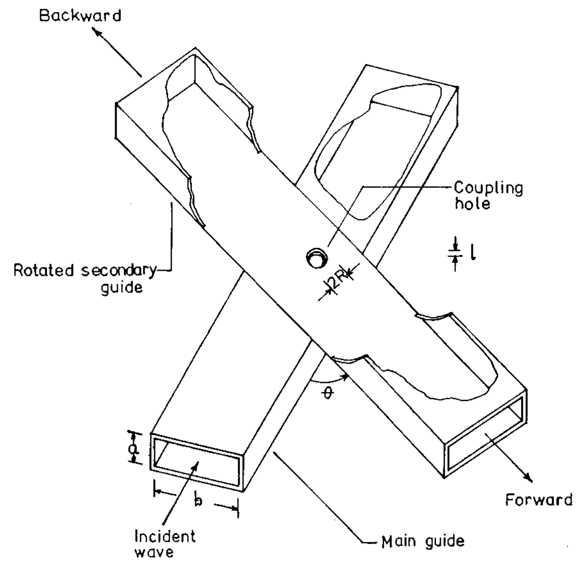


FIG. 1. General view of a Bethe coupler.

where $R$ and $T$ are the reflection and, respectively, transmission coefficients. They will be determined, together with $A$ and $B$, by requiring the continuity of the wave function and of its derivative at $z=0$ and $z=l$. The calculations are straightforward and one obtains

$$T = \frac{e^{-ikl}}{\cosh\kappa l + i[(\kappa^2-k^2)/2\kappa k]\sinh\kappa l}. \tag{11}$$

According to (6) the ratio of the transmitted energy to the incident energy is $|T|^2$. We apply this result to the Bethe coupler in the next section.

## III. THE DIRECTIVITY AND THE COUPLING FACTOR OF THE BETHE COUPLER

A general view of the Bethe coupler is given in Fig. 1. The two identical waveguides have a rectangular cross section of sides $a$ and $b$; for the main guide it extends over $-a<x<0$ and $0<y<b$. We assume $b>a$, such that a $TE_{01}$ is excited, of frequency $\omega$, and wave vector $k$ along the guide axis, related by

$$\omega^2/c^2 = k^2 + (\pi/b)^2. \tag{12}$$

The radiation passes from the main guide into the secondary one through a small cylindrical hole of radius $R$ and length $l$, bored in the common wall of the guides ($l$ is the double thickness of the wall) at $x=0$, $y=b/2$, $z=0$. The passage of radiation proceeds by the two most accessible modes excited in the cylindrical hole, $TM_{10}$ and $TE_{11}$; their frequencies are given by $\omega_{0m,e} = 2\pi c/\alpha_{m,e}R$, where $\alpha_m \cong 3.4$ and $\alpha_e \cong 1.6$ (see, for example, Refs. [4,5]), and, therefore, they are much higher than $\omega$. Accordingly, the wave vectors of the radiation propagating along the cylindrical hole are purely imaginary, being given by

$$\omega^2/c^2 = -\kappa_{m,e}^2 + \omega_{0m,e}^2/c^2. \tag{13}$$

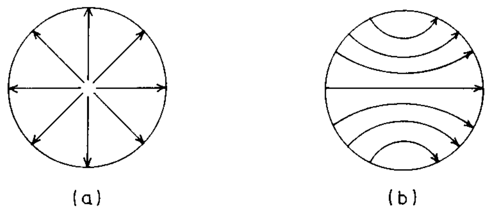


FIG. 2. Sketch of the distribution of the transverse electric field in a cylindrical hole for the $TM_{10}$ mode (a) and the $TE_{11}$ mode (b).

The cylindrical hole may be viewed as a waveguide, the radiation entering it from the main guide having the wave vector $k$ (directed along the $x$ axis) given by (12). If we denote the wave function in the main guide by $\Psi_{\rm in}$ and the wave function transmitted into the secondary guide by $\Psi_{\rm out}$, then we have for the two modes

$$\Psi_{{\rm out}\, m,e}=T_{m,e}\Psi_{\rm in}, \tag{14}$$

where the transmission coefficients $T_{m,e}$ are given by (11) with $\kappa_{m,e}$ being given by (13). The $TM_{10}$ mode has an isotropic distribution of transverse fields, shown schematically in Fig. 2(a). Consequently, the emerging wave function $\Psi_{{\rm out}\, m}$ contributes equally to the forward and backward wave functions in the secondary guide. On the contrary, the $TE_{11}$ mode has the electric field directed along the axis of the main guide at the hole center [as shown in Fig. 2(b)]. Consequently, it will contribute $\Psi_{{\rm out}\, e}\cos\theta$ to the forward wave function and $\Psi_{{\rm out}\, e}\cos(\pi-\theta)$ to the backward wave function, so that we may write

$$\Psi_f=(T_m+T_e\cos\theta)\Psi_{\rm in},$$

$$\Psi_b=(T_m-T_e\cos\theta)\Psi_{\rm in}, \tag{15}$$

by making use of (14). Since the energy density and the energy flux are proportional to $|\Psi_{f,b}|^2$, we get straightforwardly the directivity

$$D=20\log_{10}\left|\frac{1+x\cos\theta}{1-x\cos\theta}\right|, \tag{16}$$

where

$$x=T_e/T_m=\frac{\cosh\kappa_m l+i[(\kappa_m^2-k^2)/2\kappa_m k]\sinh\kappa_m l}{\cosh\kappa_e l+i[(\kappa_e^2-k^2)/2\kappa_e k]\sinh\kappa_e l}. \tag{17}$$

Making use of $x=|x|^{i\varphi}$ it is easy to find out that the directivity $D$ given by (16) has a maximum:

$$D_{\rm max}=20\log_{10}|\cot(\varphi/2)| \tag{18}$$

for $\cos\theta=1/|x|$ if $|x|>1$; if $|x|<1$ the maximum is obtained for $\theta=0$ and has the value

$$D_{\rm max}=10\log_{10}\left(\frac{1+2|x|\cos\varphi+|x|^2}{1-2|x|\cos\varphi+|x|^2}\right), \tag{19}$$

which becomes $D_{\rm max}\approx 40|x|\cos\varphi$ for $|x|\ll 1$.

Similarly, the coupling factor given by (2) is obtained as

$$C=-10\log_{10}\left|\frac{\pi R^2}{ab}(T_m+T_e\cos\theta)\right|, \tag{20}$$

where (15) has been used; we recall here that $\Psi_{\rm in}$ in (15) is related to the wave function of the radiation entering the main guide by the square root of the area ratio of the two corresponding cross sections, as seen in (20).

## IV. DISCUSSION

The wavelength $\lambda$ of the radiation in a Bethe coupler is of the order of the transverse size of the waveguides and much longer than the radius $R$ of the hole. In this case, according to (12) and (13) we have $k/\kappa_{m,e}\ll 1$ and $x$ given by (17) may be approximated by

$$x\approx\frac{\kappa_m}{\kappa_e}\,\frac{\sinh\kappa_m l-2i(k/\kappa_m)\cosh\kappa_m l}{\sinh\kappa_e l-2i(k/\kappa_e)\cosh\kappa_e l}. \tag{21}$$

Under these circumstances we may distinguish three limiting cases. The first corresponds to

$$\kappa_{m,e}l\sim l/R\ll R/\lambda\ll 1,$$

for which

$$x\approx 1+\frac{il}{2k}(\kappa_m^2-\kappa_e^2)=1+i\frac{2\pi^2 l}{kR^2}\left(\frac{1}{\alpha_m^2}-\frac{1}{\alpha_e^2}\right). \tag{22}$$

Since $|x|\geq 1$, the directivity reaches its maximum value

$$D_{\rm max}\cong 20\log_{10}\left(\frac{kR^2/\pi^2 l}{1/\alpha_e^2-1/\alpha_m^2}\right) \tag{23}$$

for $\theta\cong 0$, according to the discussion given in the preceding section. The second case corresponds to

$$R/\lambda\ll l/R\ll 1,$$

when

$$x\cong\frac{\kappa_m^2}{\kappa_e^2}\left[1-2i\frac{k}{l}\left(\frac{1}{\kappa_m^2}-\frac{1}{\kappa_e^2}\right)\right]=\frac{\alpha_e^2}{\alpha_m^2}\left[1-i\frac{kR^2}{2\pi^2 l}(\alpha_m^2-\alpha_e^2)\right]. \tag{24}$$

The maximum directivity is given in this case by (19),

$$D_{\rm max}\cong 20\log_{10}\left(\frac{\alpha_m^2+\alpha_e^2}{\alpha_m^2-\alpha_e^2}\right), \tag{25}$$

and is reached for $\theta=0$. Finally, the third case is given by

$$R/\lambda\ll 1\ll l/R$$

and

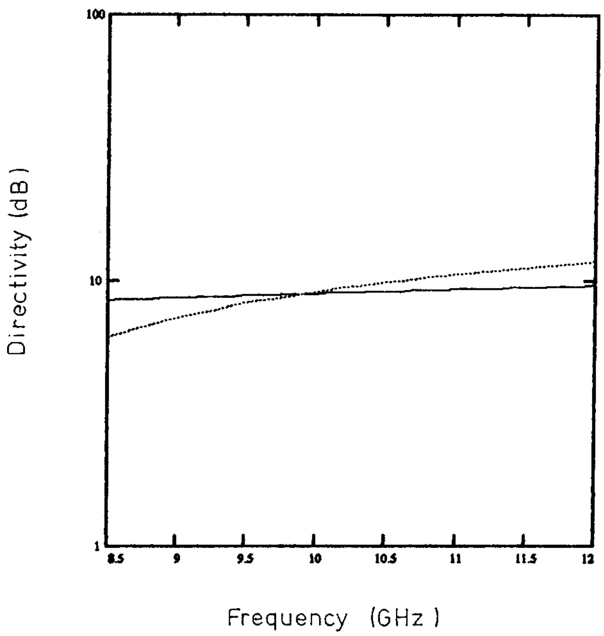


FIG. 3. The directivity given by (16) for $x$ given by (17) ($D$, solid line) and for $x_d$ given by (28) ($D_d$, dotted line) vs frequency for $l=0.28$ mm, $R=1.5$ mm, $b=2.28$ cm, and $\theta=42°$.

$$x \cong (\kappa_m/\kappa_e) e^{(\kappa_m-\kappa_e)l}[1-2ik(1/\kappa_m-1/\kappa_e)]$$
$$\cong (\alpha_e/\alpha_m) e^{-(2\pi l/R)(1/\alpha_e-1/\alpha_m)}[1-i(kR/\pi)(\alpha_m-\alpha_e)]; \tag{26}$$

the directivity attains its maximum value

$$D_{\max} \cong 40 \frac{\alpha_e}{\alpha_m} e^{-(2\pi l/R)(1/\alpha_e-1/\alpha_m)} \tag{27}$$

for $\theta=0$.

For realistic values of the parameters of the Bethe coupler and of frequencies we are in intermediate situations, and not in the limiting cases discussed above. For example, for $l=0.28$ mm, $R=1.5$ mm, $b=2.28$ cm, and $\omega=2\pi\times 10$ GHz we find out that the directivity given by (16) and (17) reaches its maximum for $\theta=42°$.

As is well known [4,5], the classical description of the Bethe coupler assumes that $x$ in (17) is given by

$$x_d = e^{(\kappa_e-\kappa_m)l}[1-(\pi c/b\omega)^2]. \tag{28}$$

The directivity $D$ given by (16) and (17), and the directivity $D_d$ corresponding to $x_d$ instead of $x$ in (16) are shown in Fig. 3 versus frequency for $l=0.28$ mm, $R=1.5$ mm, $b=2.28$ cm.

In Fig. 4 the coupling factor $C$ given by (20) is compared with the classical coupling factor [4,5]

$$C_d = -20 \log_{10} \frac{\pi d^3}{3ab\lambda}\left[\cos\theta + \frac{1}{2}(\lambda/\lambda_0)^2 \frac{F_E}{F_H}\right], \tag{29}$$

where

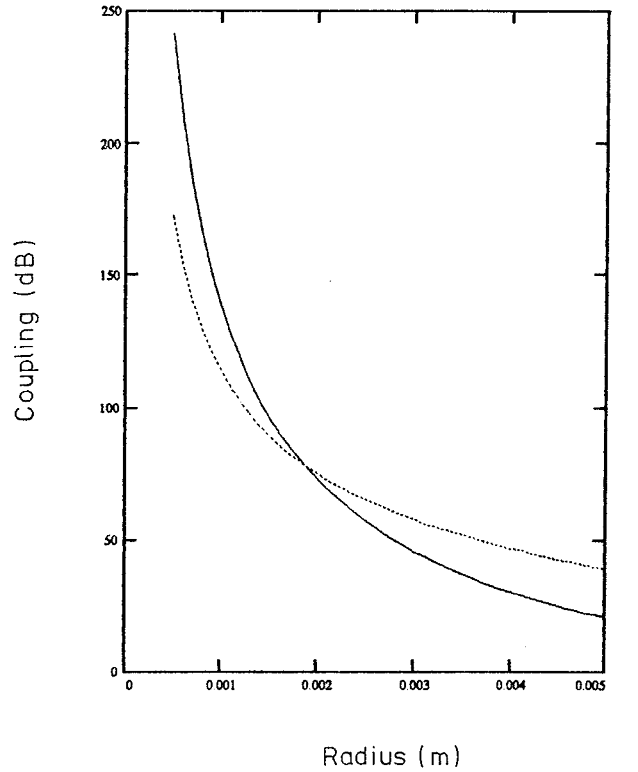


FIG. 4. The coupling $C$ given by (20) (solid line) and $C_d$ corresponding to (29) and (30) (dotted line) as functions of the radius of the circular hole for $l=2.54$ mm, $b=2.28$ cm, and $\theta=42°$, and $\omega=2\pi\times 9.6$ GHz.

$$F_E = e^{-2\pi[(1/1.31d)^2-1/\lambda_0^2]},$$
$$F_H = e^{-2\pi[(1/1.71d)^2-1/\lambda_0^2]}, \tag{30}$$

with $d=2R$ for $l=2.54$ mm, $b=2.28$ cm, $\theta=42°$, and $\omega=2\pi\times 9.6$ GHz. One can see that the present results are in good agreement with the classical ones for large values of the hole radius and high frequencies, as expected from the classical theory of the Bethe coupler. Particularly interesting is the slow variation of the maximum value of the directivity computed here over a relatively wide range of frequencies, as shown in Fig. 3 (solid line), as compared with the corresponding classical quantity shown in Fig. 3 by the dotted line. In addition, one may remark that the coupling computed here acquires higher values than the classical ones for smaller holes, as those encountered in realistic situations.

In conclusion, one may say that the propagation of the electromagnetic waves through a waveguide can be described in a very convenient manner by using the well-known quantum-mechanical concepts. The availability of simple equivalent representations for the waveguides allows a fast and accurate analysis. Various applications of the wave-function method, such as the analysis of the evanescent mode waveguide bandpass filter, the waveguide sandwhich filter, etc. are in progress.

## ACKNOWLEDGMENTS

The author is indebted to Professor M. Apostol and Dr. F. Despa for many useful discussions.

[1] H. A. Bethe (unpublished).
[2] H. A. Bethe, Phys. Rev. **66**, 163 (1944).
[3] Lord Rayleigh, Philos. Mag. **44**, 28 (1987).
[4] R. E. Collin, *Foundations for Microwave Engineering* (McGraw-Hill, New York, 1966).
[5] G. G. Montgomery, *Technique of Microwave Measurement* (McGraw-Hill, New York, 1951).
[6] N. Marinescu, Can. J. Phys. **68**, 10 (1990).
[7] N. Marinescu, R. Nistor, and M. Duma, Arch. Elektrotechnik. **75**, 155 (1992).
[8] N. Marinescu and M. Apostol, Z. Naturforsch. Teil A **47**, 935 (1992).
[9] N. Marinescu, Prog. Quantum Electron. **16**, 3 (1992).
[10] T. Martin and R. Landauer, Phys. Rev. A **45**, 2611 (1992).
[11] A. Ranfagni, D. Mugnai, and A. Agresti, Phys. Lett. A **175** 334 (1993).
[12] A. Enders and G. Nimtz, Phys. Rev. E **43**, 632 (1993).